

# Bridging a gap in Kalman filtering output estimation with correlated noises or direct feed-through from process noise into measurements

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**Abstract**—Traditional statements of the celebrated Kalman filter algorithm focus on the estimation of state, but not the output. For any outputs, measured or auxiliary, it is usually assumed that the posterior state estimates and known inputs are enough to generate the minimum variance output estimate, given by  $y_{n|n} = Cx_{n|n} + Du_n$ . Same equation is implemented in most popular control design toolboxes. It will be shown that when measurement and process noises are correlated, or when the process noise directly feeds into measurements, this equation is no longer optimal, and a correcting term of  $Hw_{n|n} \doteq H\mathbb{E}(w_n|z_n)$  is needed in above output estimation. This natural extension can allow designer to simplify noise modeling, reduce estimator order, improve robustness to unknown noise models as well as estimate unknown input, when expressed as an auxiliary output. This is directly applicable in motion-control applications which exhibits such feed-through, such as estimating disturbance thrust affecting the accelerometer measurements. Based on a proof of suboptimality [1], this correction has been accepted and implemented in Matlab 2016 [2].

## I. INTRODUCTION

Consider a discrete-time system in the canonical form with time-update equations at step  $n$  as below:

$$\begin{aligned} x_{n+1} &= Ax_n + Bu + Gw \\ y &= Cx_n + Du + Hw \\ z &= C_mx_n + D_mu + H_mw + v \end{aligned} \quad (1)$$

where  $x$  is the state vector,  $u$  is a known input vector,  $w$  is unknown input or process noise,  $z$  are measured outputs affected by measurement noise  $v$  and  $y$  is the vector of auxiliary outputs (which may well contain as a subset the measured outputs without the measurement noise). The process and measurement noises are white with following correlations.  $\mathbb{E}(ww^T) = Q$ ,  $\mathbb{E}(vv^T) = R$  and  $\mathbb{E}(wv^T) = N$ .

### A. Problem of Output Estimation

The problem we wish to solve is to compute minimum-variance estimate of the state  $x_n$ , output  $y_n$  and step-ahead prediction  $x_{n+1}$ , given a history of measurements  $z_i, i = n, n-1, n-2, \dots$ . This can be denoted by the short-hand  $x_{n|n}$ ,  $y_{n|n}$  and  $x_{n+1|n}$  respectively. The state estimation and prediction part of this problem is at the heart of linear real-time estimation of dynamic systems and was solved exactly by Kalman [3], and is described in many texts (e.g. see [4], [5], [6], [7], [8]) with varying degree of generality of assumptions. The most general form, described for example

in Kailath et al. [7] has  $G = I$ ,  $H_m = 0$  but allows arbitrary cross-correlation between  $w$  and  $v$  which makes it equivalent with (1) without loss of generality.

Yet there is a gap in all but one of above references. They do not explicitly describe the equations for optimal output estimation  $y_{n|n} \doteq \mathbb{E}(y_n|z_n)$ . The only reference from those surveyed, which describes the output equation is Kwakernaak, Sivan [4] (see eq. 4-228, Thm. 4.7), and the output estimate equation is given by:

$$y_{n|n} = Cx_{n|n} + Du_n \quad (2)$$

This is proved for uncorrelated measurement and process noises, but a footnote states that the same can be proved for correlated noises, and that  $\text{var}(y - y_{n|n}) = H'_m Q H_m + R + H_m N + N H_m$ , and innovations  $y - y_{n|n}$  are white, when  $y_{n|n}$  is computed as above and when the output vector is same as measurements without the measurement noise,  $y \doteq z - v$ .

The steady-state or time-varying Kalman filter implementations in popular control design toolboxes such as Matlab 2015b [9], Labview [10], Mathematica [11], Maple [12] and Octave-forge [13] do allow us to pose the most general problem (1) with correlated noises or feed-through and also allow us to generate estimates for both states  $x_{n|n}$  and outputs  $y_{n|n}$ . But as recently as 2015, all above implementations use the equation (2) for updating outputs. As a specific example, equation for *current* form of Kalman filter per the documentaion in [9] is:

$$\begin{bmatrix} x_{n|n} \\ y_{n|n} \\ x_{n+1|n} \end{bmatrix} = \begin{bmatrix} I - K_g C_m & -K_g D_m \\ C - CK_g C_m & D - CK_g D_m \\ A - M_{A,G} C_m & B - M_{A,G} D_m \end{bmatrix} \cdot \begin{bmatrix} x_{n|n-1} \\ u_n \end{bmatrix} + \begin{bmatrix} K_g \\ CK_g \\ M_{A,G} \end{bmatrix} z_n \quad (3)$$

where  $M_C = CK_g$  and  $M_{A,G} = AK_g + GK_{g2}$ .  $K_g = P_{n|n-1} C'_m (C_m P_{n|n-1} C'_m + \bar{R})^{-1}$ ,  $K_{g2} = (QH'_m + N)(C_m P_{n|n-1} C'_m + \bar{R})^{-1}$ ,  $\bar{R} = R + H_m Q H_m^T + H_m N + N^T H_m^T$  and  $P_{n|n-1}$  is the solution of Riccati difference iteration or the Riccati equation. Despite the complex form, it is easy to see that eq. (3) matches eq. (2). This paper proposes a change only to the output update equation for  $y_{n|n}$ .

As will be shown later in this paper, all claims from [4] mentioned above hold except the optimality, and equation (2) fails to be optimal when there is a direct feed-through, which means either  $H_m$  or  $N$  is non-zero. In such case the correction needed is:

$$y_{n|n} = Cx_{n|n} + Du_n + H\mathbb{E}(w_n|z_n) \quad (4)$$

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In section VI, we will prove the suboptimality of eq. (4) over eq. (2) through a counter-example. The loss of optimality occurs because the conditional expectation of process noise  $w_n$  knowing measurements  $z_n$  is non-zero, as they are both correlated due to direct feed-through via  $H_m$  or  $N$ , and must be accounted for to achieve best estimation of output  $y$ . Another way to see this is in frequency domain. If (1) is a discretization of a continuous time system, then  $x$  is bound to be smooth by physics, and so is the estimate  $x_{n|n}$ . But outputs  $y$  is not smooth and would not have high-frequency roll-off, due to direct feed-through in  $H$  or  $H_m$ . But using eq. (2) would constraint  $y_{n|n}$  to be smooth even though  $y$  is not. Thus estimate error is bound to be large at high frequencies. This can be avoided using the measurements  $z_n$  in a better way, in the computation of  $y_{n|n}$ , as per eq. (16) to follow.

### B. Problem of Unknown Input Estimation

Note that after substituting  $C = 0$ ,  $D = 0$  and  $H = I$  in (1), we have  $y = w$  and the problem of unknown input estimation becomes a specific case of output estimation. Traditional methods of unknown input modelling include augmenting state-space to include noise states. Such methods work well for smoothly varying unknown inputs, but due to the smoothness requirement on noise states evolution, they cannot model broad-band process noise which have significant energy at high frequencies or those which have a hard floor in frequency instead of a roll-off. Moreover, an estimator designed with a smooth noise model can lack robustness to unmodeled spikes in unknown inputs at high frequencies. Especially for an application like a wind turbine, direct feed-through exists from unknown wind thrust to sensors like accelerometers. Due to pockets of localized wind gusts and tower dam effect, the thrust on blades can have a very broad-band spectrum extending to high-frequencies with peaks at multiples of blade passing frequency. The method to improve output estimation, proposed in this paper, has a fortunate side-effect of improving the unknown input estimation, and a direct way to model broad-band noise without expanding state-space.

The paper is organized as follows. In sec. II, we derive the discrete time minimum variance estimator or the Kalman filter from the first principles, with a specific goal of state as well as output estimation. Note that we only derive state update equation for the sake of completeness, and propose no change in them compared to prior art. The only change proposed is in output update equations. The comparison with earlier approaches for output estimation is done in sec. III. A simple numerical example which demonstrates the improved estimation is described in sec. VI.

## II. DERIVATION OF DISCRETE TIME KALMAN FILTER

This section derives the formulae for a time-varying discrete-time Kalman filter from first principles. In doing so, we will leverage a key result from conditional probability of a bi-variate Gaussian distribution repeatedly.

### A. A result from conditional bivariate Gaussian distribution

Let us denote a normal distribution by its mean and variance,  $\mathcal{N}(\text{mean}, \text{variance})$ . Assume two normal random variables  $x_1 = \mathcal{N}(m_1, P_{11})$  and  $x_2 = \mathcal{N}(m_2, P_{21})$  with cross-covariance  $\text{cov}(x_1, x_2) = \mathbb{E}((x_1 - m_1)(x_2 - m_2)^T) = P_{12}$ . Then it is well-known [14] that condition distribution of  $x_1$  knowing  $x_2 = z_2$  is  $x_{1|2} = \mathcal{N}(m_{1|2}, P_{1|2})$  where

$$\begin{aligned} m_{1|2} &= \mathbb{E}(x_1 | x_2 = z_2) \\ &= \mathbb{E}(x_1) + \text{cov}(x_1, x_2) \text{var}(x_2, x_2)^{-1} (z_2 - \mathbb{E}(x_2)) \\ &= m_1 + P_{12} P_{22}^{-1} (z_2 - m_2) \end{aligned} \quad (5)$$

$$\begin{aligned} P_{1|2} &= \text{var}(x_1 | x_2 = z_2) \\ &= \text{var}(x_1, x_2) - \text{cov}(x_1, x_2) \text{var}(x_2, x_2)^{-1} \text{cov}(x_1, x_2)' \\ &= P_{11} - P_{12} P_{22}^{-1} P_{12}^T \end{aligned} \quad (6)$$

### B. State and Output update rules

At the  $n$ 'th step of filter update, we know prior state estimate  $x_{n|n-1}$  (with variance  $P_{n|n-1}$ ), measurements  $z_n$  and known inputs  $u_n$ . Tabulating cross-covariances and variances will help us in upcoming development.

$$\begin{aligned} \text{cov}(x_{n|n-1}, z_{n|n-1}) &= P_{n|n-1} C_m^T \\ \text{cov}(y_{n|n-1}, z_{n|n-1}) &= C P_{n|n-1} C_m^T + H Q H_m^T + H N \\ \text{cov}(x_{n+1|n-1}, z_{n|n-1}) &= A P_{n|n-1} C_m^T + G Q H_m^T + G N \\ \text{var}(x_{n|n-1}) &= P_{n|n-1} \\ \text{var}(z_{n|n-1}) &= C_m P_{n|n-1} C_m^T + \bar{R} \\ \text{var}(x_{n+1|n-1}) &= A P_{n|n-1} A^T + G Q G^T \end{aligned}$$

where  $\bar{R} = R + H_m Q H_m^T + H_m N + N^T H_m^T$ . Using above expressions and equation (5), by variable substitution, posterior expectations and variances of various random variables can be readily computed. The posterior expectation of state is given by (measurement update for state) :

$$\begin{aligned} \mathbb{E}(x_{n|n}) &= \mathbb{E}(x_{n|n-1} | z_{n|n-1} = z_n) \\ &= \mathbb{E}(x_{n|n-1}) + \text{cov}(x_{n|n-1}, z_{n|n-1}) \cdot \\ &\quad \text{var}(z_{n|n-1})^{-1} (z_n - \mathbb{E}(z_{n|n-1})) \\ &= x_{n|n-1} + P_{n|n-1} C_m^T \cdot \\ &\quad \left( C_m P_{n|n-1} C_m^T + \bar{R} \right)^{-1} (z_n - (C_m x_{n|n-1} + D_m u_n)) \end{aligned} \quad (7)$$

Similarly, the posterior expectation of output is given by (measurement update for outputs):

$$\begin{aligned} \mathbb{E}(y_{n|n}) &= \mathbb{E}(y_{n|n-1} | z_{n|n-1} = z_n) \\ &= \mathbb{E}(y_{n|n-1}) + \text{cov}(y_{n|n-1}, z_{n|n-1}) \cdot \\ &\quad \text{var}(z_{n|n-1})^{-1} (z_n - \mathbb{E}(z_{n|n-1})) \\ &= C x_{n|n-1} + D u_n + \left( C P_{n|n-1} C_m^T + H Q H_m^T + H N \right) \cdot \\ &\quad \left( C_m P_{n|n-1} C_m^T + \bar{R} \right)^{-1} (z_n - (C_m x_{n|n-1} + D_m u_n)) \end{aligned} \quad (8)$$

and the posterior expectation of predicted state becomes (measurement update for predicted state):

$$\begin{aligned} \mathbb{E}(x_{n+1|n}) &= \mathbb{E}(x_{n+1|n-1} | z_{n|n-1} = z_n) \\ &= \mathbb{E}(x_{n+1|n-1}) + \text{cov}(x_{n+1|n-1}, z_{n|n-1}) \cdot \\ &\quad \text{var}(z_{n|n-1})^{-1} (z_n - \mathbb{E}(z_{n|n-1})) \\ &= A x_{n|n-1} + B u_n + \left( A P_{n|n-1} C_m^T + G Q H_m^T + G N \right) \cdot \\ &\quad \left( C_m P_{n|n-1} C_m^T + \bar{R} \right)^{-1} (z_n - (C_m x_{n|n-1} + D_m u_n)) \end{aligned} \quad (9)$$

and the posterior variance for predicted state using eq. (6) is given by the discrete Riccati differential equation below:

$$P_{n+1|n} = \text{var}(x_{n+1|n-1}) - \text{cov}(x_{n+1|n-1}, z_{n|n-1}) \cdot \text{var}(z_{n|n-1})^{-1} \text{cov}(x_{n+1|n-1}, z_{n|n-1})^T \quad (10)$$

$$= (AP_{n|n-1}A^T + GQG^T) - (AP_{n|n-1}C_m^T + GQH_m^T + GN) \cdot (C_mP_{n|n-1}C_m^T + \bar{R})^{-1} (AP_{n|n-1}C_m^T + GQH_m^T + GN)^T$$

Similarly the posterior variance for predicted outputs is:

$$\text{var}(y_{n|n}) = \text{var}(y_{n|n-1}) - \text{cov}(y_{n|n-1}, z_{n|n-1}) \cdot \text{var}(z_{n|n-1})^{-1} \text{cov}(y_{n|n-1}, z_{n|n-1})^T \quad (11)$$

$$= (CP_{n|n-1}C^T + HQH^T) - (CP_{n|n-1}C_m^T + HQH_m^T + HN) \cdot (C_mP_{n|n-1}C_m^T + \bar{R})^{-1} (CP_{n|n-1}C_m^T + HQH_m^T + HN)^T$$

In summary, after each measurement  $z_n$ ,  $x_{n|n-1}$  and  $P_{n|n+1}$  are propagated forward in time to generate  $x_{n+1|n}$  and  $P_{n+1|n}$  and recursively there on, through equations (9) and (10). Estimates of outputs and states,  $y_{n|n}$  and  $x_{n|n}$  based on Kalman filter are also generated by equations (8) and (7).

**Remark.** Note that if estimated outputs are same as measured, i.e. when  $C = C_m$ ,  $H = H_m$ ,  $N = 0$ , eq. (11) simplifies to

$$\text{var}(y_{n|n}) = (C_mP_{n|n-1}C_m^T + H_mQH_m^T) \cdot (C_mP_{n|n-1}C_m^T + H_mQH_m^T + R)^{-1} \cdot R \preceq R \quad (12)$$

Thus the posterior output estimate is more accurate than the measurement, which matches the intuition.

### III. COMPARISON WITH THE PRIOR ART

Equations (7), (8) and (9) can be summarized in following matrix equation.

$$\begin{bmatrix} x_{n|n} \\ y_{n|n} \\ x_{n+1|n} \end{bmatrix} = \begin{bmatrix} I - K_g C_m & -K_g D_m \\ C - M_{C,H} C_m & D - M_{C,H} D_m \\ A - M_{A,G} C_m & B - M_{A,G} D_m \end{bmatrix} \cdot \begin{bmatrix} x_{n|n-1} \\ u_n \end{bmatrix} + \begin{bmatrix} K_g \\ M_{C,H} \\ M_{A,G} \end{bmatrix} z_n \quad (13)$$

where  $M_{C,H} = CK_g + HK_{g2}$  and  $M_{A,G} = AK_g + GK_{g2}$ .  $K_g = P_{n|n-1}C_m'(C_mP_{n|n-1}C_m' + \bar{R})^{-1}$  and  $K_{g2} = (QH_m' + N)(C_mP_{n|n-1}C_m' + \bar{R})^{-1}$ .

Using (13), it can be shown that  $y_{n|n}$  and  $x_{n|n}$  are related in following way.

$$y_{n|n} = Cx_{n|n} + Du_n + HK_{g2} \cdot (z_n - (C_mx_{n|n-1} + D_mu_n)) \quad (14)$$

Comparing eq. (14) with eq. (2) used in widely used design tools, we see that there needs to be an additional correction needed in computing  $y_{n|n}$  when process noise feeds into measurements, and traditional equality  $y_{n|n} = Cx_{n|n} + Du_n$  would not hold. This is so, because conditional expectation

of process noise, knowing the measurement with its feed-through, is non-zero.

$$w_{n|n} \doteq \mathbb{E}(w_n|z_n) = K_{g2} \cdot (z_n - (C_mx_{n|n-1} + D_mu_n)) \quad (15)$$

and eq. (14) can be expressed as

$$y_{n|n} = Cx_{n|n} + Du_n + H\mathbb{E}(w_{n|n}) \quad (16)$$

Note that in absence of the feed-through and correlation between process and measurement noise ( $H_m = 0$  and  $N = 0$ ),  $K_{g2}$  would be zero and (14) reverts to previous implementation (2). Thus this correction only affects cases with correlated process and measurement noises or direct-feed through.

Note that the state-update and prediction equations from eq. (13) exactly matches with those from previous references and implementations. Thus the only modification proposed here is in computation of the output estimate,  $y_{n|n}$ .

### IV. STEADY-STATE KALMAN FILTER

Above time-varying update equations simplify considerably for a steady-state Kalman filter. The variance update equation (10) simplifies to the Riccati equation.

$$P = (APA^T + GQG^T) - (APC_m^T + GQH_m^T + GN) \cdot (C_mPC_m^T + \bar{R})^{-1} (APC_m^T + GQH_m^T + GN)^T \quad (17)$$

and the update equations (13) become linear time-invariant once  $P$  is fixed, as  $K_g$ ,  $K_{g2}$  and all the coefficients become constant, and estimates evolve as per following dynamical system.

$$x_{n+1|n} = (A - M_{A,G} C_m)x_{n|n-1} + \begin{bmatrix} M_{A,G} \\ B - M_{A,G} D_m \end{bmatrix} \begin{bmatrix} z_n \\ u_n \end{bmatrix}$$

$$\begin{bmatrix} x_{n|n} \\ y_{n|n} \end{bmatrix} = \begin{bmatrix} I - K_g C_m \\ C - M_{C,H} C_m \end{bmatrix} \cdot x_{n|n-1} + \begin{bmatrix} K_g \\ M_{C,H} \end{bmatrix} z_n + \begin{bmatrix} -K_g D_m \\ D - M_{C,H} D_m \end{bmatrix} \cdot \begin{bmatrix} z_n \\ u_n \end{bmatrix} \quad (18)$$

### V. REMARKS ON CONTINUOUS TIME ESTIMATION

As seen above, for the equations for discrete time Kalman filter, the only correction needed is in the formula for  $y_{n|n}$  in eqs. (14) and (13) to compute minimum variance output estimate. The same correction carries over to the continuous time system. In continuous time systems, the state update is continuous, while measurements may be take at variable times. Thus prior state estimate  $x_{n|n-1}$  is discrete time systems is replaced by prior estimate  $\hat{x}_t^-$  and it's variance would be  $\hat{P}_t^-$ , and we can use the second equation from eq. (13) to derive continuous-time analogue of optimal posterior output estimate,  $\hat{y}_t^+$ .

$$\hat{y}_t^+ = C\hat{x}_t^- + Du_t + (CK_g + HK_{g2}) \cdot (z_t - (C_m\hat{x}_t^- + D_mu_t)) \quad (19)$$

where  $K_g = P_t^- C_m'(C_mP_t^- C_m' + \bar{R})^{-1}$ ,  $K_{g2} = (QH_m' + N)(C_mP_t^- C_m' + \bar{R})^{-1}$ , which is same as the discrete time analog, except that  $P_t^-$  is the prior variance of the prior state estimate  $\hat{x}_t^-$ .

## VI. A NUMERICAL EXAMPLE

Consider a simple example of a system with a direct feed-through.

$$\dot{x} = -0.1x + 2w, \quad y = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w, \quad z = y(1) + v$$

$$Q = \text{var}(w) = 1, \quad R = \text{var}(v) = 0.1, \quad N = \text{cov}(w, v) = 0$$

Note that the first output  $y(1)$  has feed-through from process noise  $w$  and is measured with certain noise as  $z$  and the second output  $y(2) = w$  is an estimate of unknown input expressed as an auxiliary output.

Kalman estimators of three kinds were implemented on discretized version of above problem with time-step 0.1 seconds. These were:

- Time varying Kalman filter as per eq. (13), denoted by legend *new*.
- Steady state Kalman filter as per eq. (18), denoted by legend *new ss*.
- Steady state Kalman filter using `kalman.m` function in Matlab [9] controls toolbox, denoted by legend *prev ss*.

### A. Nominal performance

The estimates of measured output  $y(1)$  by three methods are compared in fig. 1, which also compares the estimate errors. It is clear that method (c) fails to capture the high frequencies present in the measured output, and only captures contribution due to state, which evolves smoothly. The estimate errors variances using both methods (a) and (b) is 0.0910, which matches eq. (12). This is 10 times lower than method (c) whose error variance is 0.99, which is close to  $H_m Q H_m^T + R$  as claimed in [4]. The estimates from methods (a) and (b) are nearly identical after the initial transient. This matches with the common intuition that time-varying Kalman filter rapidly converges to steady-state Kalman filter, as the value of state variance  $P_n$  approaches Riccati solution (17).

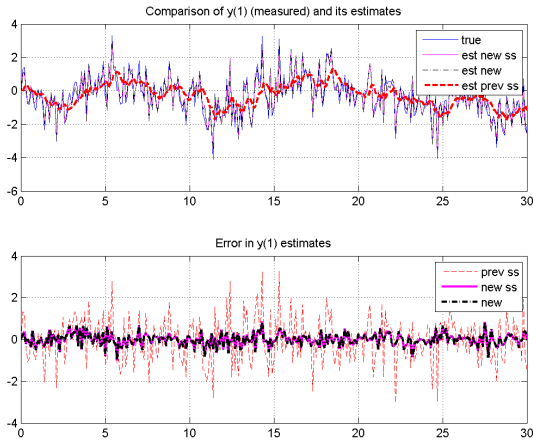


Fig. 1. Measured output estimation  $y(1)$

Fig. 2 compared the unknown input estimates by three methods, and again shows much better performance by the new method in unknown input estimation.

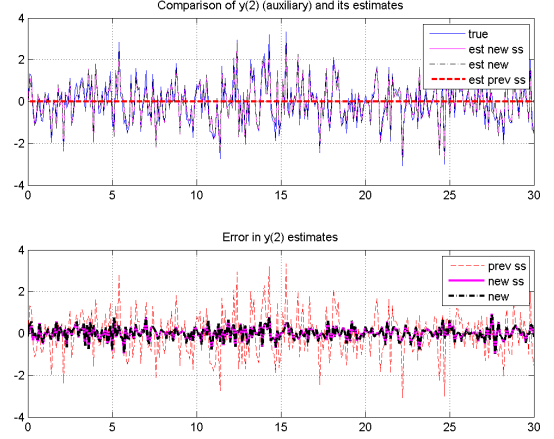


Fig. 2. Unknown input estimation  $y(2)$

Fig. 3 compares the state estimates by three methods, and they show near-perfect match, which shows that state estimation by new method per eq. (9) is done in exactly same manner as method (c).

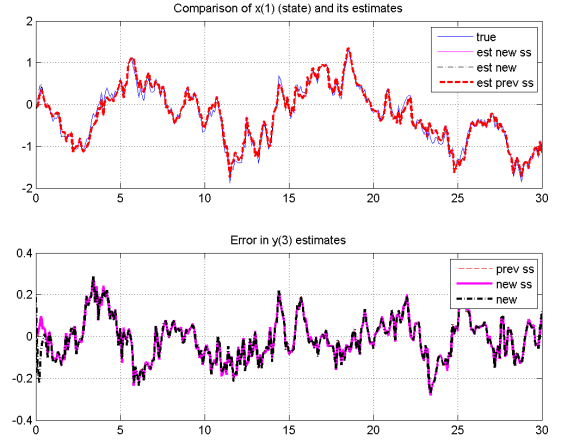


Fig. 3. State estimation  $x(1)$

### B. Robustness to unmodeled bias in disturbance

Usually it is hard to guarantee robustness of optimal estimator to unmodeled disturbance dynamics. Still, as an example, the three estimators above were tested against an unmodeled random walk drift in addition to the white noise in the disturbance  $w$ . The `true` signal in fig. 5 shows this disturbance. Fig. 4 compares  $y(1)$  estimation and shows great improvement by new estimators over previous method (c), which has a clear drift in estimate error. Fig. 5 compares  $y(2)$  or unknown input estimation. Since  $y(2) = 0 \cdot x + w$ , the estimate from method (c) is zero and estimate error is

large. On the same problem, the new method estimates drift and high frequency disturbance quite well.

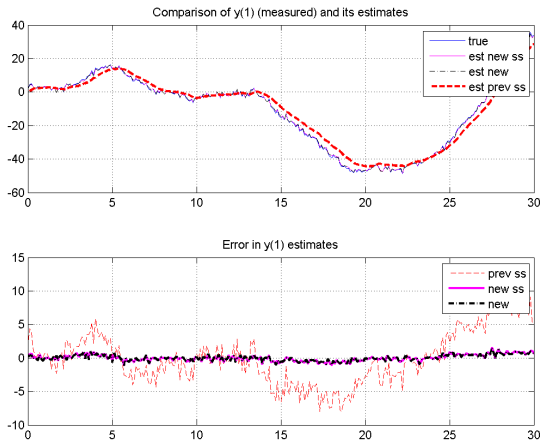


Fig. 4. Measured output  $y(1)$  estimation with unmodeled bias in process noise

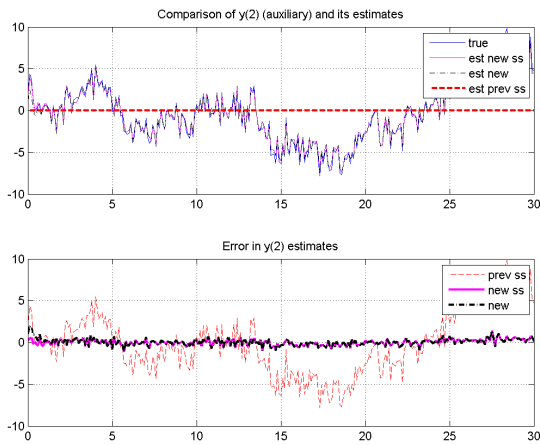


Fig. 5. Unknown input  $y(2)$  estimation with unmodeled bias in process noise

## VII. CONCLUDING REMARKS

In summary, until now, due to a gap in output estimate update rules in Kalman filter and sub-optimal implementation within widely used design tools ([9], [10], [11], [12], [13]) per eq. (2), designers had no way of accommodating direct feed-through of even a part of the process noise into measurements, except by expanding state-space to include noise states, which may or may not be physical. But this workaround left much to be desired as it was constrained by smoothness requirements on the unknown input estimates and thus incurred large estimation errors at high frequencies and reduced the estimation bandwidth for such outputs. Bandwidth could only be increased at the cost of simplicity or robustness due to increased model order or new tuning parameters.

The proposed simple correction to output estimates per eq. (14), fills this gap as it makes use of posterior estimate of the unknown input computed from measurement which contain its feed-through. Thus it guarantees minimum variance output estimate. The performance of estimator has been demonstrated through a simple numerical example. The robustness to unmodeled inputs (e.g. bias or drift) seems to have greatly improved by this correction.

Such estimator is directly useful for the problems of estimating thrust from accelerometers mounted on flexible structures, especially if the thrust is broad-band and hard to model physically. The paper also makes the case for correcting the implementations in commonly used toolboxes for control system design, as all the surveyed implementations/documentations suffer from the sub-optimality in the output estimation for problems with direct feed-through or correlated process and measurement noises.

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